Intégrales doubles

$\mathbb{R}^2$ euclidien, orienté
Bon directe

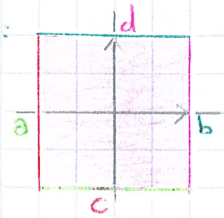
I Compact élémentaire

$D \subset \mathbb{R}^2$ compact élémentaire $\Leftrightarrow \exists (a,b,c,d) \in \mathbb{R}^4$, $a<b$, $c<d$

$\begin{cases} \varphi_1, \varphi_2 : [ab] \to \mathbb{R} \\ \psi_1, \psi_2 : [cd] \to \mathbb{R} \end{cases}$ continues

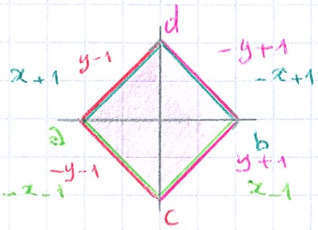
$\begin{cases} \forall x \in [ab], & \varphi_1(x) < \varphi_2(x) \\ \forall y \in [cd], & \psi_1(y) < \psi_2(y) \\ \forall (x,y) \in \mathbb{R}^2, & (x,y) \in D \Leftrightarrow \begin{cases} a \leq x \leq b \\ \varphi_1(x) \leq y \leq \varphi_2(x) \end{cases} \Leftrightarrow \begin{cases} c \leq y \leq d \\ \psi_1(y) \leq x \leq \psi_2(y) \end{cases} \end{cases}$

ex :



$\varphi_1 : \left| \begin{array}{l} [-1,1] \to \mathbb{R} \\ x \mapsto -1 \end{array} \right.$ $\quad \psi_1 : \left| \begin{array}{l} [-1,1] \to \mathbb{R} \\ y \mapsto -1 \end{array} \right.$

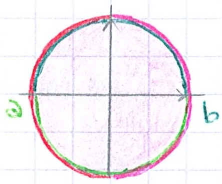
$\varphi_2 : \left| \begin{array}{l} [-1,1] \to \mathbb{R} \\ x \mapsto 1 \end{array} \right.$ $\quad \psi_2 : \left| \begin{array}{l} [-1,1] \to \mathbb{R} \\ y \mapsto 1 \end{array} \right.$



$a = -1$, $b = 1$ $\quad c = -1$, $d = 1$

$\varphi_1 : \left| \begin{array}{l} [-1,1] \to \mathbb{R} \\ x \mapsto |x| - 1 \end{array} \right.$ $\quad \psi_1 : \left| \begin{array}{l} [-1,1] \to \mathbb{R} \\ y \mapsto |y| - 1 \end{array} \right.$

$\varphi_2 : \left| \begin{array}{l} [-1,1] \to \mathbb{R} \\ x \mapsto -|x| + 1 \end{array} \right.$ $\quad \psi_2 : \left| \begin{array}{l} [-1,1] \to \mathbb{R} \\ y \mapsto -|y| + 1 \end{array} \right.$



$a = -1$, $b = 1$ $\quad c = -1$, $d = 1$

$\varphi_1 : \left| \begin{array}{l} [-1,1] \to \mathbb{R} \\ x \mapsto -\sqrt{1-x^2} \end{array} \right.$ $\quad \psi_1 : \left| \begin{array}{l} [-1,1] \to \mathbb{R} \\ y \mapsto -\sqrt{1-y^2} \end{array} \right.$

$\varphi_2 : \left| \begin{array}{l} [-1,1] \to \mathbb{R} \\ x \mapsto \sqrt{1-x^2} \end{array} \right.$ $\quad \psi_2 : \left| \begin{array}{l} [-1,1] \to \mathbb{R} \\ y \mapsto \sqrt{1-y^2} \end{array} \right.$

ne sont pas des compacts élém

Pp 1) $D \subset [ab] \times [cd]$

2) un compact élémentaire est un compact

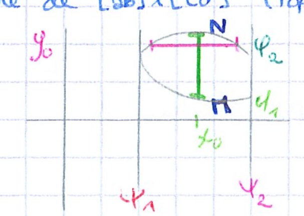
* D est borné
* D est fermé : $F : \left| \begin{array}{l} [ab] \times [cd] \to \mathbb{R}^4 \\ (x,y) \mapsto (x-a,\ b-x,\ y-\varphi_1(x),\ \varphi_2(x)-y) \end{array} \right.$

$D = \underbrace{F^{-1}}_{F \text{ continue}} \underbrace{(\mathbb{R}^+ \times \mathbb{R}^+ \times \mathbb{R}^+ \times \mathbb{R}^+)}_{\text{fermé}}$ $\quad D$ fermé de $[ab] \times [cd]$ (topo relative)

3) $\forall x_0 \in [ab]$, $D \cap \{(x_0, y) / y \in \mathbb{R}\}$ est un segment

$\forall y_0 \in [cd]$, $D \cap \{(x, y_0) / x \in \mathbb{R}\}$

$[(x_0, \varphi_1(x_0)), (x_0, \varphi_2(x_0))] = [MN]$



II Intégration

D compact élém, $f : D \to \mathbb{K}$ continue $\quad (\mathbb{K} = \mathbb{R}$ ou $\mathbb{C})$

$\hat{f} : \left| \begin{array}{l} \mathbb{R}^2 \to \mathbb{K} \\ (x,y) \mapsto \begin{cases} f(x,y) & \text{si } (x,y) \in D \\ 0 & \text{si } (x,y) \notin D \end{cases} \end{array} \right.$ $\quad \hat{f}$ est à support compact : $\overline{\hat{f}^{-1}(\mathbb{K}^*)} \subset [ab] \times [cd]$

a priori, $\hat{f}$ n'est pas continue $\quad (x,y) / (x,y) \in D$ + l'adhérence

Hh $\quad \forall x \in \mathbb{R}$, $\left| \begin{array}{l} \mathbb{R} \to \mathbb{K} \\ y \mapsto \hat{f}(x,y) \end{array} \right.$ est continue, intégrable

Fubini pr les fcts intégrables sur un rect

$\left| \begin{array}{l} \mathbb{R} \to \mathbb{K} \\ x \mapsto \int_{-\infty}^{+\infty} \hat{f}(x,y)\, dy \end{array} \right.$ est continue, intégrable

$\forall y \in \mathbb{R}$, $\begin{vmatrix} \mathbb{R} \longrightarrow \mathbb{K} \\ x \longmapsto \hat{f}(x,y) \end{vmatrix}$ est continue, intégrable

$\begin{vmatrix} \mathbb{R} \longrightarrow \mathbb{K} \\ y \longmapsto \int_{-\infty}^{+\infty} \hat{f}(x,y)\,dx \end{vmatrix}$ est continue, intégrable

$$\int_{-\infty}^{+\infty}\left(\int_{-\infty}^{+\infty} \hat{f}(x,y)\,dy\right)dx = \int_{-\infty}^{+\infty}\left(\int_{-\infty}^{+\infty} \hat{f}(x,y)\,dx\right)dy = \iint_D f(x,y)\,dx\,dy$$

$$\text{aire de } D = \iint_D \mathbb{1}\,dx\,dy$$

* calcul :

* si D est le pavé $[ab]\times[cd]$: (Fubini sur un pavé)

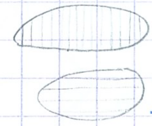
$$\iint_D f = \int_{-\infty}^{+\infty}\left(\int_{-\infty}^{+\infty}\hat{f}(x,y)\,dy\right)dx = \int_{-\infty}^{+\infty}\Big(\underbrace{\int_c^d \hat{f}(x,y)\,dy}_{\text{nul ailleurs que sur } [cd]}\Big)dx = \int_a^b\left(\int_c^d \hat{f}(x,y)\,dy\right)dx = \int_a^b\left(\int_c^d f(x,y)\,dy\right)dx$$

$y\in[cd] \Rightarrow x \in [ab]$

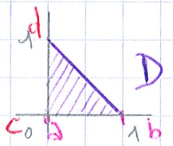
* si D compact élém. :

$$\iint_D f = \int_{-\infty}^{+\infty}\left(\int_{-\infty}^{+\infty}\hat{f}(x,y)\,dy\right)dx = \int_a^b\left(\int_{-\infty}^{+\infty}\hat{f}(x,y)\,dy\right)dx = \int_a^b\left(\int_{\varphi_1(x)}^{\varphi_2(x)}\hat{f}(x,y)\,dy\right)dx = \int_a^b\left(\int_{\varphi_1(x)}^{\varphi_2(x)} f(x,y)\,dy\right)dx$$

de m, $\displaystyle\iint_D f = \int_c^d\left(\int_{\psi_1(y)}^{\psi_2(y)} f(x,y)\,dx\right)dy$



ex : $f : \begin{vmatrix} D \longrightarrow \mathbb{R} \\ (x,y) \longmapsto (x^2+y^2) \end{vmatrix}$,

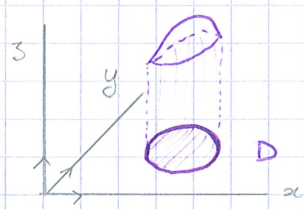


$$\iint_D f = \int_0^1\left(\int_0^{1-x}(x^2+y^2)\,dy\right)dx = \int_0^1\left[x^2y+\frac{y^3}{3}\right]_0^{1-x}dx = \int_0^1\left(x^2(1-x)+\frac{(1-x)^3}{3}\right)dx$$

$$= \left[-\frac{x^4}{4}+\frac{x^3}{3}+\frac{1}{3}\cdot\frac{-1}{4}(1-x)^4\right]_0^1 = -\frac{1}{4}+\frac{1}{3}-\left(-\frac{1}{12}\right) = \frac{1}{6}$$

rem : une intégrale dble représente un volume :

$$\sum_{i,j}(x_{i+1}-x_i)(y_{j+1}-y_j)\,f(\xi_i,y_i)$$

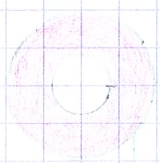


III Généralisation

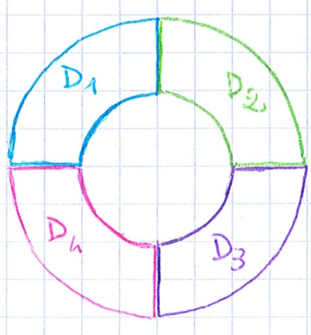
compact simple du plan : réunion finie de compacts élém. d'intérieur 2 à 2 disjoints

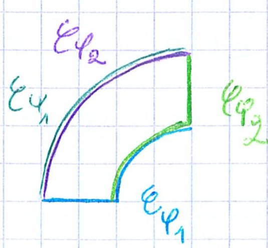
$K = \bigcup_{1\leq i\leq n} D_i$ / $\begin{cases} D_i \text{ compact élém} \\ i\neq j,\ \mathring{D}_i\cap\mathring{D}_j=\emptyset \end{cases}$

ex :



compact simple





(l'image d'un compact simple par f continue n'est pas forcément un compact simple)

K compact simple du plan $K = \bigcup D_i$ une décomposit°

$f : K \to \mathbb{K}$ continue on pose $\displaystyle\iint_K f = \sum_{i=1}^{n}\iint_{D_i} f$ intérieur de f sur K

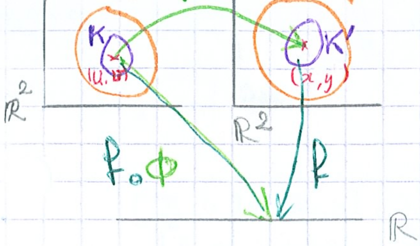
Pp) ne dépend pas de la décomposit°

IV Changements de coordonnées

U, V ouverts de $\mathbb{R}^2$, $\Phi\in C^1(U,\mathbb{R}^2)$, $\Phi(U)\subset V$

si $\begin{cases} K \text{ compact simple de } \mathbb{R}^2,\ \Phi(K)=K' \\ \Phi\ \ C^1\text{-difféo de } \mathring{K} \text{ sur } \mathring{K}' \\ f\in C^0(V,\mathbb{R}) \end{cases}$ alors $\displaystyle\iint_{K'} f = \iint_{\Phi(K)} f = \iint_K f\circ\Phi\,|\mathrm{jac}\,\Phi|$

sert à tenir compte du chgt d'orientat° induit par Φ



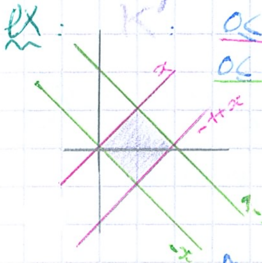
autrement dit,

$$\Phi : \begin{vmatrix} U \to V \\ \begin{pmatrix} u \\ v \end{pmatrix} \longmapsto \begin{pmatrix} x \\ y \end{pmatrix} = \Phi\begin{pmatrix} u \\ v \end{pmatrix} = \begin{pmatrix} \varphi_1(u,v) \\ \varphi_2(u,v) \end{pmatrix} = \begin{pmatrix} x(u,v) \\ y(u,v) \end{pmatrix} \end{vmatrix}$$

$$\text{Jac}\,\Phi = \begin{pmatrix} \frac{\partial \varphi_1}{\partial u}(u,v) & \frac{\partial \varphi_1}{\partial v}(u,v) \\ \frac{\partial \varphi_2}{\partial u}(u,v) & \frac{\partial \varphi_2}{\partial v}(u,v) \end{pmatrix}$$

$$\iint_{K'} f = \iint_{K'} f(x,y)\,dx\,dy = \iint_{\Phi(K)} \underbrace{f(\varphi_1(u,v), \varphi_2(u,v))}_{f\circ\Phi} \left| \begin{vmatrix} \frac{\partial \varphi_1}{\partial u} & \frac{\partial \varphi_1}{\partial v} \\ \frac{\partial \varphi_2}{\partial u} & \frac{\partial \varphi_2}{\partial v} \end{vmatrix} \right| du\,dv = \iint_{\Phi(K)} f(x(u,v), y(u,v)) \left| \begin{vmatrix} \frac{\partial x}{\partial u} & \frac{\partial x}{\partial v} \\ \frac{\partial y}{\partial u} & \frac{\partial y}{\partial v} \end{vmatrix} \right| d$$

ex : K' : $\underline{0 \leq x-y \leq 1}$, $\underline{0 \leq x+y \leq 1}$ compact simple $f : \begin{vmatrix} K' \to \mathbb{R} \\ (x,y) \longmapsto x^2 - y^2 \end{vmatrix}$



$\begin{cases} u = x-y \\ v = x+y \end{cases}$ $\begin{cases} x = \frac{u+v}{2} \\ y = \frac{-u+v}{2} \end{cases}$ $\Phi : \begin{vmatrix} \mathbb{R}^2 \longrightarrow \mathbb{R}^2 \\ \begin{pmatrix} u \\ v \end{pmatrix} \longmapsto \begin{pmatrix} \frac{u+v}{2} \\ \frac{-u+v}{2} \end{pmatrix} \end{vmatrix}$ $\Phi \in Gl(\mathbb{R}^2)$ difféo

$$I = \iint_{K'} f(x,y)\,dx\,dy = \iint_{\Phi(K)} f = \iint_K f\circ\Phi(u,v)\,|\text{jac}\,\Phi(u,v)|\,du\,dv$$

$K = \Phi^{-1}(K')$ $\begin{cases} 0 \leq u \leq 1 \\ 0 \leq v \leq 1 \end{cases}$ Φ C^∞-difféo de $\mathring{K}$ $\begin{pmatrix} 0<u<1 \\ 0<v<1 \end{pmatrix}$ sur $\mathring{K}'$: $\begin{cases} 0 < x-y < 1 \\ 0 < x+y < 1 \end{cases}$

$\text{Jac}\,\Phi = \text{mat}\,\Phi = \begin{pmatrix} 1/2 & 1/2 \\ -1/2 & 1/2 \end{pmatrix}$

$\text{jac}\,\Phi = \frac{1}{2} > 0$

$f\circ\Phi(u,v) = f\left(\frac{u+v}{2}, \frac{-u+v}{2}\right) = \left(\frac{u+v}{2}\right)^2 - \left(\frac{-u+v}{2}\right)^2$

$$I = \iint_K uv \left|\frac{1}{2}\right| du\,dv = \frac{1}{2} \iint_{[0,1]\times[0,1]} uv\,du\,dv = \frac{1}{2}\int_0^1 u\,du \int_0^1 v\,dv = \frac{1}{8}$$

rem : $K = [0,1]$, $\Phi : \begin{vmatrix} \mathbb{R} \longrightarrow \mathbb{R} \\ u \longmapsto au = x \end{vmatrix}$ $a \in \mathbb{R}^*$, $K' = \Phi(K) = [0,a]$ $f : \begin{vmatrix} \mathbb{R} \to \mathbb{R} \\ x \longmapsto x \end{vmatrix}$

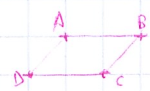
Φ C^∞-difféo de $\mathring{K}$ sur $\mathring{K}'$ $\text{Jac}\,\Phi = (a)$, $\text{jac}\,\Phi = a$

$$\int_{K'} f(x)\,dx = \int_K f\circ\Phi(u)\,|\text{jac}\,\Phi|\,du$$

si $a > 0$: $\int_0^a f(x)\,dx = \frac{a^2}{2} = \int_0^1 au\,|a|\,du = \frac{1}{2}a|a|$

si $a < 0$: $\int_a^0 f(x)\,dx = -\frac{a^2}{2} = \int_0^1 au\,|a|\,du = \frac{1}{2}a|a|$

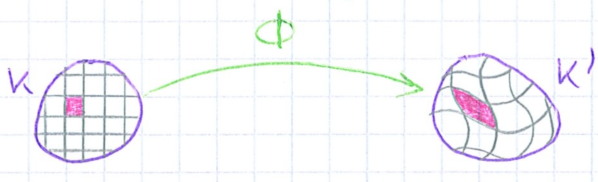
* K parallélogramme, Φ affine bijective, $K' = \Phi(X)$ parallélogramme



plus petit convexe qui contient A,B

$f : \begin{vmatrix} \mathbb{R}^2 \longrightarrow \mathbb{R} \\ \begin{pmatrix} x \\ y \end{pmatrix} \longmapsto 1 \end{vmatrix}$ $\iint_{K'} 1\,dx\,dy = \iint_K 1 \cdot |\text{jac}\,\Phi|\,du\,dv$

aire géométrique de K' = (aire géométrique de K) $\times$ $|\text{jac}\,\Phi|$



passage en coordonnées polaires

$U = \mathbb{R}^2$, $V = \mathbb{R}^2$ $\Phi : \begin{vmatrix} \mathbb{R}^2 \longrightarrow \mathbb{R}^2 \\ \begin{pmatrix} \rho \\ \theta \end{pmatrix} \longmapsto \begin{pmatrix} \rho\cos\theta = x \\ \rho\sin\theta = y \end{pmatrix} \end{vmatrix}$ $K \subset \mathbb{R}^+ \times [-\pi, \pi]$ compact simple de U

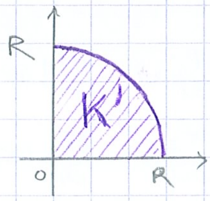
donc $\mathring{K} \subset \mathbb{R}^{+*} \times]-\pi, \pi[$

Φ C^∞-difféo de $\mathring{K}$ sur $\mathring{K}' = \mathring{\Phi(K)}$

si K' est un compact simple de $\mathbb{R}^2$, $\iint_{K'} f(x,y)\,dx\,dy = \iint_K f(\rho\cos\theta, \rho\sin\theta)\,\rho\,d\rho\,d\theta$

ex : $K' = \{(x,y) \in \mathbb{R}^2 \,/\, 0 \leq x, 0 \leq y, x^2+y^2 \leq R^2\}$ $\quad R>0$ $\quad \text{jac}\,\Phi = \begin{vmatrix} \cos\theta & -\rho\sin\theta \\ \sin\theta & \rho\cos\theta \end{vmatrix} = \rho > 0$

$f : \left| \begin{array}{l} \mathbb{R}^2 \longrightarrow \mathbb{R} \\ \binom{x}{y} \longmapsto e^{-x^2-y^2} \end{array} \right.$

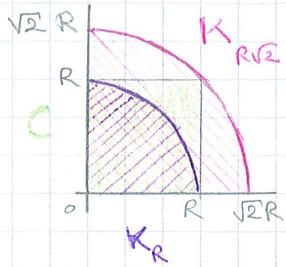


$0 \leq \rho \leq R$ $\qquad K = [0,R] \times [0, \frac{\pi}{2}]$

$0 \leq \theta \leq \frac{\pi}{2}$ $\qquad \mathring{K} =]0,R[\times]0, \frac{\pi}{2}[$

Φ C^∞ difféo de $\mathring{K}$ sur $\mathring{K}'$

$$\iint_{K'} f(x,y)\,dx\,dy = \iint_K e^{-\rho^2}\rho\,d\rho\,d\theta = \int_0^{\pi/2} d\theta \int_0^R e^{-\rho^2}\rho\,d\rho = \frac{\pi}{2}\left[-\frac{e^{-\rho^2}}{2}\right]_0^R = \frac{\pi}{4}(1-e^{-R^2})$$



$C = [0,R] \times [0,R]$

$$\iint_C f(x,y)\,dx\,dy = \iint_{[0,R]^2} e^{-x^2}e^{-y^2}\,dx\,dy = \left(\int_0^R e^{-x^2}dx\right)^2$$

$\begin{cases} K'_R \subset C_R \subset K'_{R\sqrt{2}} \\ f \text{ positive} \end{cases}$

$$\iint_{K'_R} f \leq \iint_{C_R} f \leq \iint_{K'_{R\sqrt{2}}} f$$

$$\Rightarrow \frac{\pi}{4}(1-e^{-R^2}) \leq \left(\int_0^R e^{-x^2}dx\right)^2 \leq \frac{\pi}{4}(1-e^{-2R^2})$$

$R \to +\infty$: $\frac{\pi}{4} = \left(\int_0^{+\infty} e^{-x^2}dx\right)^2$ d'où $\int_0^{+\infty} e^{-x^2}dx = \frac{\sqrt{\pi}}{2}$